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ELASTIC EQUILIBRIUM OF A WEDGE WITH A CRACK

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There is examined the plane homogeneous problem of elasticity theory on the equilibrium of an infinite wedge in whose bisector plane there is an inner rectilinear semi-infinite crack. The wedge faces and the crack edges are load-free. The stresses tend to zero at infinity but their principal vector and principal moment differ from zero and are given by a condition. The problem is reduced to solving a Wiener — Hopf functional equation by using a Mellin transformation. An exact solution of the equation is given and the stress intensity coefficient at the crack vertex is calculated.

1. Formulation of the problem. Let us consider the equilibrium of an infinite elastic wedge with aperture angle 2α $(0 < \alpha \leqslant \pi)$, in whose bisector plane there is an infinite crack for $y = 0$, $x > l$ (Fig. 1). The wedge faces and crack edges are load-free. The stresses tend to zero at infinity, but their principal vector and principal moment differ from zero and equal $(0, Y)$ and M, respectively.

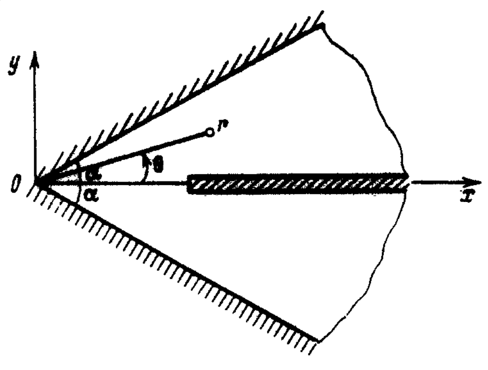

Fig. 1

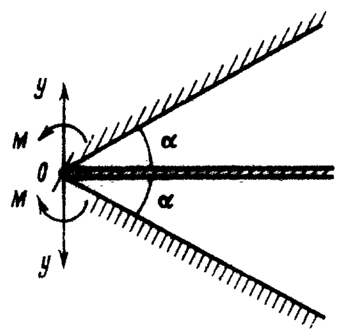

Fig. 2

The ligament $y = 0$, $0 < x < l$ between the lower $-\alpha < \theta < 0$ and upper $0 < \theta < \alpha$ wedge therefore transmits the given force $(0, Y)$ and the given moment M.

The boundary conditions of the problem have the form

$$\theta = \alpha, \quad \sigma_\theta = \tau_{r\theta} = 0 \tag{1.1}$$

$$\theta = 0, \quad \tau_{r\theta} = 0$$

$$\theta = 0, \quad r < l, \quad \frac{\partial^2 u_\theta}{\partial r^2} = 0 \tag{1.2}$$

$$\theta = 0, \quad r > l, \quad \sigma_\theta = 0$$

(σ_θ, $\tau_{r\theta}$, σ_r are the stresses, and u_θ, u_r the displacements).

Note. The normal displacement u_θ, and not just its second derivative with respect to the radius is zero, as is written in the first condition of (1.2), as follows from the continuity of the displacements at the ligament and from the symmetry of the problem. However, the stresses in a problem with the condition $\theta = 0, r < l$, $u_\theta = 0$ will be the same as the stresses in a problem with the condition $\theta = 0, r < l$, $\partial^2 u_\theta / \partial r^2 = 0$ (when all the remaining conditions are retained).

By assumption, the relationships

$$\int_0^l \sigma_\theta(r, 0)\, dr = Y, \quad \int_0^l \sigma_\theta(r, 0)\, r\, dr = M \tag{1.3}$$

are valid.

Information will later be required about the roots of the equation

$$\Delta \equiv \sin 2p\alpha + p \sin 2\alpha = 0 \tag{1.4}$$

(p is a complex number) in the strip $0 < \operatorname{Re} p < 1$.

It is known that (1.4) has no roots in the mentioned strip for $0 < \alpha < \pi / 2$ [1].

Lemma. For any α $(\pi / 2 < \alpha < \pi)$, the equation $\Delta(p) = 0$ has a unique root in the domain $0 < \operatorname{Re} p < 1$, $\operatorname{Im} p \geqslant 0$. This root is real and belongs to the interval $(^1/_2, 1)$.

We shall not present the proof of the lemma which is based on passing from (1.4) to an appropriate real system and using the apparatus of differential calculus.

The problem under consideration is a homogeneous, singular problem of elasticity theory whose singularities are the crack vertex, the wedge apex, and the infinitely remote point. The boundary value problem at the singularities is not defined. Additional conditions at the singularities should be formulated in the formulation of the correct singular boundary value problem. The following assertion is used to formulate these conditions [2]: The solution of the correct boundary value problem of elasticity theory behaves in an infinitesimal neighborhood of a singularity as the eigenfunction asymptotically greatest in absolute value, which corresponds to the canonical singular problem.

The passage from the initial problem to the corresponding canonical singular problems, which is realized by using the "microscope principle", and the investigation of these latter (see [2]) show that the condition

$$\theta = 0, \quad r \to l - 0, \quad \sigma_\theta \sim K_{\mathrm{I}}\, [2\pi\, (l - r)]^{-1/2} \tag{1.5}$$

$$\theta = +0, \quad r \to l + 0, \quad \frac{\partial^2 u_\theta}{\partial r^2} \sim -(1 - \nu^2)\, E^{-1} K_{\mathrm{I}}\, (2\pi)^{-1/2} (r - l)^{-3/2}$$

can be formulated at the crack apex (E is Young's modulus, ν is the Poisson's ratio, K_{I} is the stress intensity coefficient at the crack apex) and the following condition

$$\theta = 0, \quad r \to 0 \tag{1.6}$$

$$\sigma_\theta \to 0 \quad (0 < \alpha < \pi / 2)$$

$$\sigma_\theta \sim A / r^{1-\lambda} \quad (\pi / 2 < \alpha \leqslant \pi)$$

at the wedge vertex ($\lambda = \lambda(\alpha)$ is the unique root of (1.4) in the strip $0 < \operatorname{Re} p < 1$, and A is a real constant).

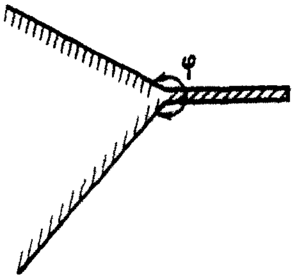

Fig. 3

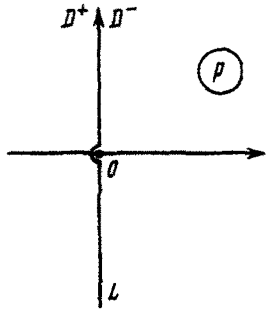

Fig. 4

We apply the microscope principle to the initial problem to formulate the conditions at infinity. Let $r/l \to \infty$. This can be realized for $r \to \infty$, $l = \text{const}$ and for $r = \text{const}$, $l \to 0$. The passage to the limit $r = \text{const}$, $l \to 0$ corresponds to the singular limit problem (see Fig. 2). Two wedges with stress-free faces, whose vertices are combined at the zero point, are shown in the figure. The force $(0, Y)$ and the moment M act at the apex of the upper wedge, and the force $(0, -Y)$ and the moment $-M$ at the apex of the lower wedge. The solution of this problem is the asymptote at infinity for the solution of the initial problem. In particular, the relation

$$\theta = +0, \quad r \to \infty \tag{1.7}$$

$$\frac{\partial^2 u_\theta}{\partial r^2} \sim -\frac{1-\nu^2}{E}\left(\frac{2\alpha+\sin 2\alpha}{\alpha^2-\sin^2\alpha}\frac{Y}{r^2} + \frac{4\cos\alpha}{\alpha\cos\alpha-\sin\alpha}\frac{M}{r^3}\right)$$

holds.

Therefore, the nontrivial solution of the homogeneous problem (1.1) and (1.2)under the additional conditions (1.5) — (1.7) at the singularities are to be found.

Without limiting the generality, the spacing between the wedge vertex and the crack apex can be considered unity.

Undeservedly little attention has been spent on the investigation of nontrivial solutions of homogeneous problems of elasticity theory. This is explained by the widespread conviction that only the trivial solution of such problems exists according to the uniqueness theorem. This latter is true for problems of the class S (the Saint-Venant principle is valid in this class of problems). A nontrivial solution of homogeneous problems [2] exists in problems of the class N (the Saint-Venant principle does not hold in this class of problems). Namely, as a rule these problems are of greatest pracitcal value. When problems of the class N must be solved, researchers ordinarily limit themselves to the construction of a unique solution of the inhomogeneous problem and neglect the form of the homogeneous solutions.

As an illustration, let us examine the known investigation of Khrapkov [3].

The problem for a wedge with a nonsymmetric rectilinear crack emerging from the wedge angle is considered by Khrapkov. The wedge angle is φ (Fig. 3). It is assumed that the wedge faces are free, while arbitrary loads are applied to the crack edges. It is assumed that the stresses tend to zero as an additional condition at infinity. This latter condition is sufficient for the existence of a unique solution just in problems.

of class S, i.e., for $\varphi < \pi$ (see [2]). In problems of the class N (i.e., for $\varphi > \pi$) the correct formulation of the problem should include giving definite coefficients whose dimensionality depends on the magnitude of the angle φ. The solution of the problem is not unique for $\varphi > \pi$ in the Khrapkov formulation since there exists a nontrivial solution of the homogeneous problem (determined to the accuracy of two arbitrary constants). As the external loads tend to zero, the Khrapkov solution tends to zero, which corresponds to the trivial solution of the homogeneous problem.

An elastic wedgelike plate $0 \leqslant r < \infty, 0 \leqslant \theta \leqslant \alpha$ of thickness h to which an elastic absolutely flexible variable-section rod is fastened at the section $0 \leqslant r \leqslant 1$, $\theta = 0$, is considered in the paper of Nuller [4]. A force T is applied to the rod endface. Tangential forces $f(r)$ act on the free part of the rod side surface at $0 \leqslant r \leqslant 1$.

If $f(r) \equiv 0$, $T = 0$ and $\alpha > \pi$ then the B. M. Nuller solution is trivial while a nontrivial solution of the homogeneous problem mentioned exists in this case. This solution is determined to the accuracy of two arbitrary constants.

This last remark also refers to the Nuller paper [5]. The wedge $-\alpha \leqslant \theta \leqslant \alpha$, $0 \leqslant r < \infty$ is considered, where one part of the boundary ($\theta = \pm\alpha$, $0 \leqslant r \leqslant 1$) is under sliding support conditions, while the other is stress free. A load equivalent to the prescribed force acts through a rigid holder (wedgelike stamp) on the wedge.

For $\alpha > \pi/2$ and $P = 0$ (P is a given arbitrary force), a nontrivial solution of the problem exists. In this case the Nuller solution is trivial.

The homogeneous problem considered in this paper belongs to the class S. Its nontrivial solution is constructed below.

2. Wiener–Hopf functional equation and its solution. Applying the Mellin transform

$$m^*(p) = \int_0^\infty m(r) r^p \, dr$$

to the equilibrium equations and the strain compatibility condition, we obtain [1]

$$\sigma_\theta^*(p, \theta) = A_1 \sin(p+1)\theta + A_2 \sin(p-1)\theta + A_3 \cos(p+1)\theta + A_4 \cos(p-1)\theta \tag{2.1}$$

$$\tau_{r\theta}^* = (p-1)^{-1} \frac{d\sigma_\theta^*}{d\theta}, \quad p\sigma_r^* = (p-1)^{-1} \frac{d^2\sigma_\theta^*}{d\theta^2} - \sigma_\theta^*$$

($A_i(p)$ are unknown functions of p ($i = 1, 2, 3, 4$)). By using the boundary conditions (1.1) we arrive at the system

$$\begin{gathered} A_1 \sin(p+1)\alpha + A_2 \sin(p-1)\alpha + A_3 \cos(p+1)\alpha + A_4 \cos(p-1)\alpha = 0 \\ A_1 (p+1)\cos(p+1)\alpha + A_2 (p-1)\cos(p-1)\alpha - A_3 (p+1)\sin(p+1)\alpha - A_4 (p-1)\sin(p-1)\alpha = 0 \\ A_1 (p+1) + A_2 (p-1) = 0 \end{gathered} \tag{2.2}$$

whose solution we write as

$$A_2 = -(p+1)(p-1)^{-1} A_1, \quad A_3 = -2(p \sin^2\alpha + \sin^2 p\alpha)\Delta^{-1} A_1 \tag{2.3}$$

$$A_4 = -2\,(p+1)\,(p-1)^{-1}\,(p\sin^2\alpha - \sin^2 p\alpha)\,\Delta^{-1}A_1$$

According to (2.1) and (2.3)

$$\sigma_\theta^*\,(p,0) = -4\,(p-1)^{-1}\,(p^2\sin^2\alpha - \sin^2 p\alpha)\,\Delta^{-1}A_1 \tag{2.4}$$

By using Hooke's law, and taking account of (2.1) and (2.3), we find

$$\int_0^\infty \frac{\partial^2 u_\theta}{\partial r^2}\bigg|_{\theta=+0} r^{p+1}\,dr = \frac{4\,(1-\nu^2)}{E}(p+1)\,(p-1)^{-1}A_1 \tag{2.5}$$

Eliminating A_1 in the relationships (2.4) and (2.5), we arrive at the fundamental Wiener — Hopf equation

$$\Phi^-(p) = -2\,(p+1)^{-1}\,(p^2\sin^2\alpha - \sin^2 p\alpha)\,\Delta^{-1}\Phi^+(p) \tag{2.6}$$

$$\Phi^-(p) = \int_0^1 \sigma_\theta(r,0)\,r^p\,dr,\quad \Phi^+(p) = \frac{E}{2\,(1-\nu^2)}\int_1^\infty \frac{\partial^2 u_\theta}{\partial r^2}\bigg|_{\theta=+0} r^{p+1}\,dr$$

which we rewrite as

$$\Phi^-(p) = (p+1)^{-1}\,\mathrm{tg}\,p\pi G(p)\,\Phi^+(p) \tag{2.7}$$
$$G(p) = -2\,\mathrm{ctg}\,p\pi\,(p^2\sin^2\alpha - \sin^2 p\alpha)\,\Delta^{-1}$$

We set

$$\lambda_0 = \begin{cases}\lambda, & \pi/2 < \alpha \leqslant \pi\\ 1, & 0 < \alpha \leqslant \pi/2\end{cases}$$

According to (1.6) and (1.7), the function $\Phi^+(p)$ is analytic in the half-plane $\mathrm{Re}\,p < 0$, and the function $\Phi^-(p)$ in the half-plane $\mathrm{Re}\,p > -\lambda_0$.

Let us examine the contour L consisting of the imaginary axis with the exception of a small symmetric section around the origin, and a left semicircle of small radius with center at the origin (Fig. 4), in the plane of the complex variable p. The direction of traversing the contour agrees with the direction of the imaginary axis. The domains on the left and right of the contour will be denoted by D^+ and D^-.

The function $G(p)$ has neither zeros nor poles on the contour L and tends to unity along it as $p \to \infty$. Therefore, the representation

$$G(p) = \frac{G^+(p)}{G^-(p)}\quad (p \in L) \tag{2.8}$$

$$\exp\left[\frac{1}{2\pi i}\int_L \frac{\ln G(t)}{t-p}\,dt\right] = \begin{cases}G^+(p), & p \in D^+\\ G^-(p), & p \in D^-\end{cases}$$

is valid.

By using the representation [6]

$$\mathrm{ctg}\,p\pi = p^{-1}K^+(p)\,K^-(p),\quad K^\pm(p) = \frac{\Gamma(1\mp p)}{\Gamma(1/2 \pm p)}$$

where the functions $K^+(p)$ and $K^-(p)$ are analytic, have no zeroes and satisfy the conditions

$$K^+(p) \sim \sqrt{-p}\quad (p\to\infty),\quad \mathrm{Re}\,p < 1/2 \tag{2.9}$$
$$K^-(p) \sim \sqrt{p}\quad (p\to\infty),\quad \mathrm{Re}\,p > -1/2$$

we obtain by taking care of (2.7) and (2.8)

$$(p+1)\,\Phi^-(p)K^-(p)G^-(p) = p\,[K^+(p)]^{-1}G^+(p)\Phi^+(p) \quad (p \in L) \qquad (2.10)$$

The function in the left side of (2.10) is analytic in the domain D^- and the function in the right side is analytic in the domain D^+. On the basis of the principle of continuous extension they equal the very same function which in analytic in the whole plane p.

Let us find this single analytic function. Starting from the relationships (1.5), we find $(p \to \infty)$

$$\Phi^-(p) \sim (2p)^{-1/2}K_I, \quad \Phi^+(p) \sim (-p/2)^{1/2}K_I \qquad (2.11)$$

On the basis of (2.8), (2.9) and (2.11) it follows from (2.10) that the single analytic function is $c_0 + c_1p$ (c_0, c_1 are constants to be determined).

Taking the conditions resulting from (1.3) into account

$$\Phi^-(0) = Y, \quad \Phi^-(1) = M$$

by using (2.10) we obtain a system to determine c_0 and c_1

$$YK^-(0)G^-(0) = c_0, \quad 2MK^-(1)\,G^-(1) = c_0 + c_1$$

Hence

$$c_0 = Y\left[\frac{2\alpha + \sin 2\alpha}{2(\alpha^2 - \sin^2\alpha)}\right]^{1/2}, \quad c_1 = 4M\pi^{-1/2}G^-(1) - c_0$$

The solution of the functional equation is written as

$$\Phi^-(p) = \frac{c_0 + c_1p}{(p+1)K^-(p)G^-(p)}, \quad \Phi^+(p) = \frac{K^+(p)(c_0 + c_1p)}{pG^+(p)} \qquad (2.12)$$

3. **Analysis of the solution.** Let us evaluate the stress intensity coefficient at the crack apex. We find

$$\Phi^-(p) \sim c_1p^{-1/2} \qquad (3.1)$$

from the first formula in (2.12) as $p \to \infty$

Comparing the asymptotics of the function $\Phi^-(p)$ in (2.11) and (3.1) and going over to dimensional variables, we obtain

$$K_I = 4\left(\frac{2}{\pi}\right)^{1/2} G^-(1)\,Ml^{-3/2} - \left(\frac{2\alpha + \sin 2\alpha}{\alpha^2 - \sin^2\alpha}\right)^{1/2} Yl^{-1/2}$$

Let us study the behavior of the stress σ_θ for $\theta = 0$, $r \to 0$.

Using the second formula of (2.12), (2.6), the lemma, the Mellin inversion formula, and the theorem of residues, we arrive at the relationships

$$\sigma_\theta(r, 0) \to 0 \quad (0 < \alpha < \pi/2)$$

$$\sigma_\theta(r, 0) \sim \frac{2(\sin^2\lambda\alpha - \lambda^2\sin^2\alpha)}{2\alpha\cos 2\lambda\alpha + \sin 2\alpha}\,\frac{\Gamma(1+\lambda)}{\Gamma(1/2+\lambda)}\,\frac{c_0 - \lambda c_1}{G^+(-\lambda)\lambda(\lambda-1)}\,r^{\lambda-1} \quad (\pi/2 < \alpha \leqslant \pi)$$

As a result of investigating the behavior of the stresses for $0 < \theta < \alpha$ and $r \to \infty$ we obtain

$$\sigma_r(r, \theta) \sim \frac{(1 - \cos 2\alpha)\cos\theta - (2\alpha + \sin 2\alpha)\sin\theta}{\sin^2\alpha - \alpha^2}\,\frac{Y}{r} +$$

$$\frac{2\sin(\alpha-2\theta)}{\sin\alpha-\alpha\cos\alpha}\frac{M}{r^2}$$

$$\tau_{r\theta}(r,\theta)\sim\frac{2\sin\theta\sin(\alpha-\theta)}{\sin\alpha-\alpha\cos\alpha}\frac{M}{r^2},\quad \sigma_\theta(r,\theta)=o\left(\frac{1}{r^2}\right)$$

As should have been expected, these formulas agree with the known solution of the corresponding canonical singular problem.

The limit cases $\alpha = \pi/2$ and $\alpha = \pi$ of the problem considered were studied earlier by other authors. [7 — 9].

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